Summary of my Master Thesis
A Category-Theoretical Approach to Fuzzy Sets
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This thesis analyzes the notion of a fuzzy set using a category-theoretical approach. A fuzzy set is a set whose elements belong to the set only with a certain degree represented by a real number in the interval [0,1]. The manipulation of fuzzy sets is studied in fuzzy logic. A category is a very general mathematical concept. The central notion of category theory is the notion of a topos that can be defined as a category that possesses a rich enough logical structure to develop or reconstruct most theories of mathematics.

An analysis of fuzzy sets in terms of categories will strengthen the foundations of the theory of fuzzy set. This is important because it brings precision in a field, which has been criticized ever since Zadeh published his first paper on fuzzy sets as being theoretically weak.

The first topic I address is related to work done by Goguen. Goguen generalized Zadeh’s original operational definition of fuzzy sets. This definition highlights the algebraic structure of fuzzy sets. More specifically, Goguen substituted the interval [0,1] by a lattice, which then turned out to be a Heyting algebra H. I show how this result provides a justification of Zadeh’s definition of a fuzzy set.

The second topic concerns another insight of Goguen. He suggested an axiomatization of fuzzy sets in terms of category theory. A more classical axiomatic approach to fuzzy sets along the lines of Zermelo and Fraenkel was previously put forward by Chapin. I critically assess these two proposals and argue that Goguen’s proposal is superior.

The third and most important topic of my thesis is a discussion of the result that the category Set(H) – with H being a Heyting algebra – for fuzzy sets defined by Goguen is a quasitopos. The category Set(H) satisfies all but one of the conditions of a topos. The condition that is not satisfied requires a topos to have a so-called subobject classifier. I emphasize that this is a serious lack for Set(H) since the subobject classifier can be roughly identified with the set of truth values.

The fourth and final topic concerns the notion of fuzzy identity, and I show that this notion is problematic. In classical set theory, two sets can be defined to be identical. In fuzzy sets theory, a fuzzy identity relation between the elements of the same set needs to be specified. Such sets are called totally fuzzy and their category Set(H) can be shown to be a topos. It was Higgs who proved this, showing that Set(H) is equivalent to the category of sheaves. If a proper identity predicate is defined, a category of fuzzy sets that is also a topos can be defined.

I conclude that fuzzy set theory is a fascinating part of mathematics with a solid foundation provided by category theory.